

Breaking Down The “Cosine Fourth Power Law”

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Why are the corners of the field of view in the image captured by a camera lens usually darker than the center? For one thing, camera lenses — by design — often introduce “vignetting” into the image, which is the deliberate clipping of rays at the corners of the field of view in order to cut away excessive lens aberrations. But, it is also known that corner areas in an image can get dark even without vignetting, due in part to the so-called “cosine fourth power law.”¹

According to this “law,” when a lens projects the image of a uniform source onto a screen, in the absence of vignetting, the illumination flux density (i.e., the optical power per unit area) across the screen from the center to the edge varies according to the fourth power of the cosine of the angle between the optic axis and the oblique ray striking the screen. Actually, optical designers know this “law” does not apply generally to all lens conditions.²⁻¹⁰ Fundamental principles of optical radiative flux transfer in lens systems allow one to tune the illumination distribution across the image by varying lens design characteristics.

In this article, we take a tour into the fascinating physics governing the illumination of images in lens systems.

Relative Illumination In Lens Systems

In lens design, one characterizes the illumination distribution across the screen where the image resides in terms of a quantity known as the lens’ *relative illumination* — the ratio of the irradiance (i.e., the power per unit area) at any off-axis position of the image to the irradiance at the center of the image. In the absence of deliberate vignetting, the relative illumination produced by a lens across the screen depends on a number of factors, including the nature of rays scattering or emitting from the source, the shape and size of the lens’ pupils, the distortion in the image, the oblique ray angles behind (and in front of) the lens, and a little-known quantity called “differential distortion.”

Let’s look at a simple example. Fig. 1 shows what’s called a rear landscape lens — a single lens element placed behind the iris (known in lens design terminology as the “aperture stop”). The object is a uniform, circular disk of diameter 200 mm, centered on the optic axis, and it emits incoherent light at a wavelength of 550 nm. Additionally, this object is assumed to be a “Lambertian” source, which is the condition that most optical design programs assume when computing the image’s relative illumination.

A Lambertian source emits (or scatters) rays in such a way that its radiance (i.e., the flux per unit source area within a unit solid angle) in any direction is constant. Perhaps an approximate example of such a source is the surface of white print paper, such as that used in laser printers and

photocopiers. The prescription for this lens system is given in Table 1, so if you have access to an optical design program, you can enter this prescription and examine this lens in the manner that I'm about to describe.

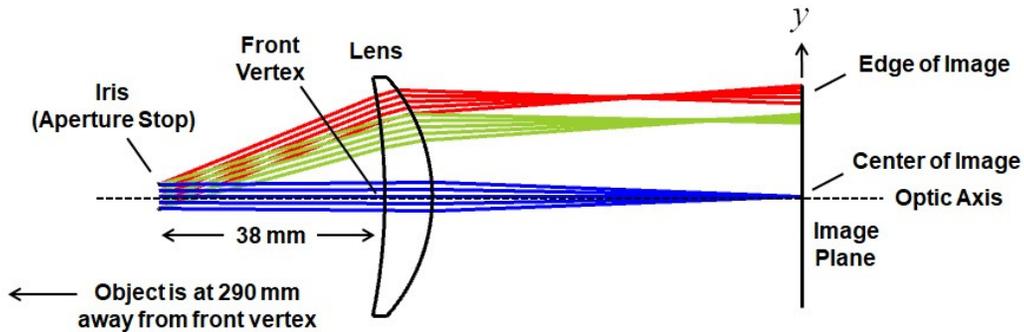


Fig. 1 — A rear landscape lens example.

Surface	Radius	Thickness	Material	Semi-Diameter
OBJECT	Infinity	252.000		100.000
1 (STOP)	Infinity	38.000		2.000
2	-100.0	8.000	N—SF6	20.000
3	-30.0	62.277		20.000
IMAGE	Infinity			18.602

Table 1 — Prescription for the lens shown in Fig. 1 (dimension units in mm)

Fig. 2a shows a plot of this lens' relative illumination, which has been computed using an optical design program.¹¹ Note that the relative illumination is conventionally plotted as a function of the image's corresponding object height. Since the object's full diameter is 200 mm, its height above the optic axis is 100 mm, which is the magnitude of the maximum along the horizontal axis in Fig. 2a. From this plot, it is quite evident that the center of the image is actually darker than the edge! Fig. 2b shows a false-color plot of the surface irradiance at the image. Note that the irradiance at the farthest edge of the image is slightly dimmed due to aberrations at the image's edge (note the large blur spot at the farthest y-position above the optic axis in Fig. 1).

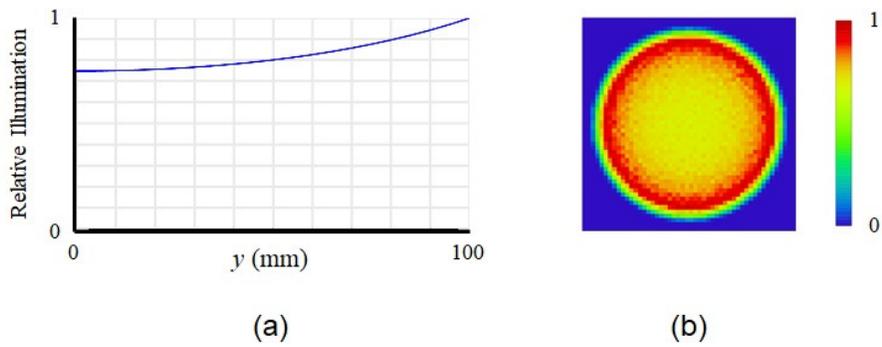


Fig. 2 — (a) Relative Illumination of the lens in Fig. 1. (b) False color representation of the relative illumination across the image.

Now, let's examine what happens to the relative illumination for this lens as a function of the position of the iris, at three different locations relative to the lens' front vertex (Figs. 3 – 5). In Fig. 3a, the iris is maintained at the same location as in Fig. 1, yielding the relative illumination shown in Fig. 3b (which is the same as in Fig. 2a).

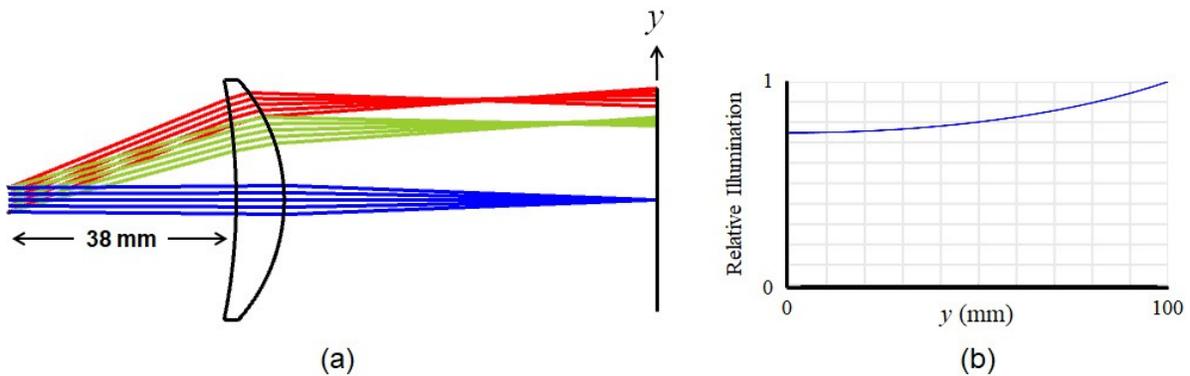


Fig. 3 — (a) Iris at 38 mm from front vertex. (b) Resulting relative illumination.

In Fig. 4a, the iris is at 29 mm from the lens's front vertex (note that the total distance from the object to the front vertex of the lens has been maintained at 290 mm), yielding the relative illumination shown in Fig. 4b.

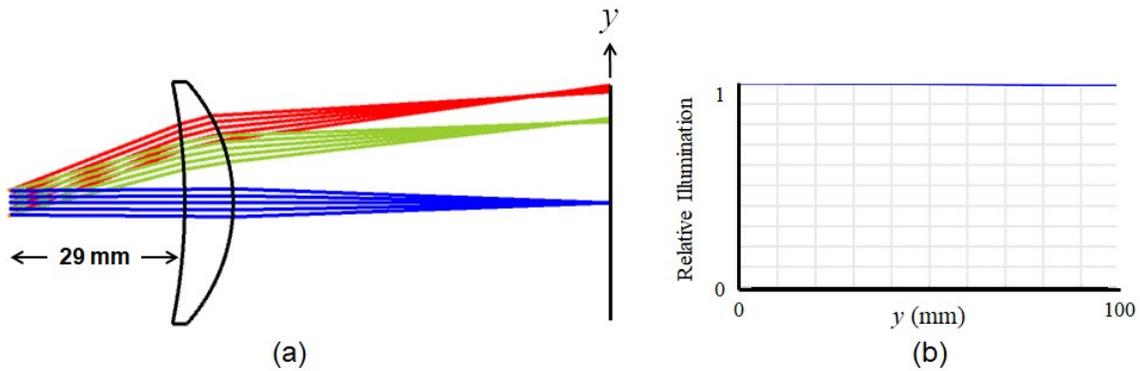


Fig. 4 — (a) Iris at 29 mm from front vertex. (b) Resulting relative illumination.

In Fig. 5a, the iris is even closer to the lens, and is at 10 mm from the lens’s front vertex (whilst again maintaining the object distance to the front vertex of the lens at 290 mm), resulting in the relative illumination plot shown in Fig. 5b. Clearly, through simple manipulation of the iris position, one can tune the relative illumination for this lens.

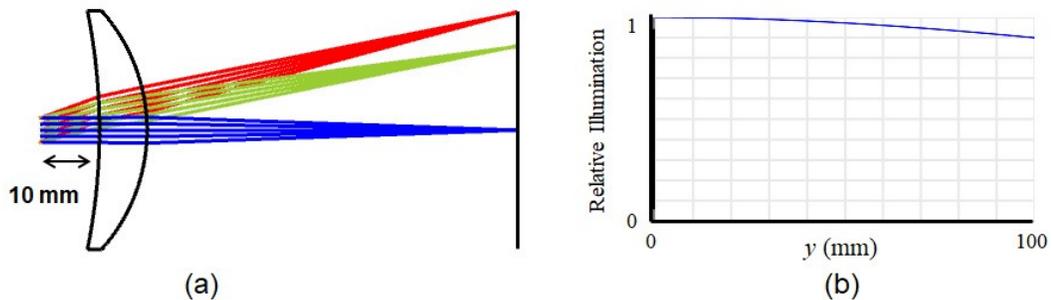


Fig. 5 — (a) Iris at 10 mm from front vertex. (b) Resulting relative illumination.

The Science Behind Relative Illumination

Let’s try to understand the results in Figs. 2 – 5 based on the science behind relative illumination. The general principle is that, as a consequence of the presence of an iris in a lens system, there results what’s called the “entrance and exit pupils” of the lens system. The entrance pupil is what the iris looks like from the perspective of the object, while the exit pupil is what the iris looks like from the perspective of the image. It is the geometry in which rays pass through the entrance and exit pupils that gives rise to the form of the relative illumination observed. The technical term for this ray geometry is the “projected solid angle.”

From the perspective of the image, as one looks at the exit pupil from the optic axis and up (i.e., in the direction of y shown in Fig. 1), lens elements situated between the iris and the image can change the apparent shape and size of the exit pupil, which can yield either lesser or greater flux passing through the exit pupil above the optic axis. Thus, whenever the flux is greater (as in Figs. 2 – 3), the relative illumination increases from the center to the edge, and when the flux is less, the relative illumination decreases. But, when the shape and size of the exit pupil match the effect of the obliquity of the rays striking the image, the relative illumination can be made constant (Fig. 4).

Optical design programs usually apply numerical algorithms to compute relative illumination based on ray geometries at the exit pupil.¹² Due to the law of conservation of energy (i.e., optical flux passing through the entrance pupil is equal to the flux passing out through the exit pupil), relative illumination can also be computed based on ray geometries at the entrance pupil. In fact, it can be shown^{10, 13} that, under the condition of having a small iris diameter (i.e., a large f /number or low numerical aperture), by applying the ray geometry shown in Fig. 6, the relative illumination $R(y)$ of a lens system may be approximated by the following equation:

$$R(y) \approx \frac{[A_{EN}(y)/A_{EN}(0)] \cos^4 \theta}{(1 + D)[1 + D + y(dD/dy)]} \quad \text{Eq. (1)}$$

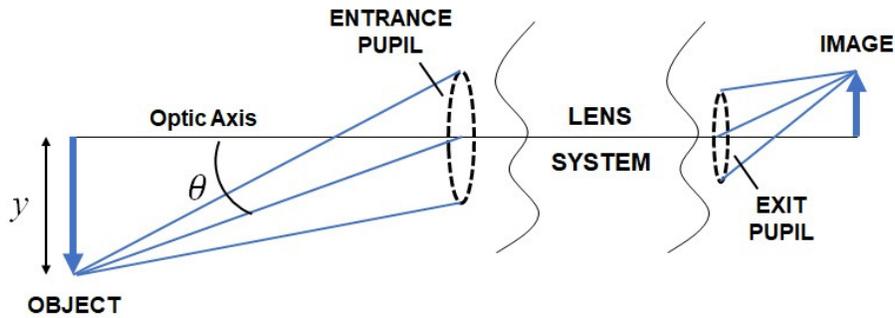


Fig. 6 — Ray geometry for Eq. 1.

In Eq. (1), the quantity $[A_{EN}(y)/A_{EN}(0)]$ is the ratio of the entrance pupil area at an off-axis point to its area at the optic axis from the perspective of the object, and D is the image distortion (e.g., if the image distortion were –10 percent, then $D = -0.1$). The quantity dD/dy has been called differential distortion^{10, 13} and it is the instantaneous rate of change of image distortion with respect to the object’s height y above the optic axis. Due to differential distortion, the relative illumination

at any position in the image can be affected by the *variation* of the image's distortion, even if the image distortion itself were zero at that position.

Let's apply Eq. (1) to the conditions shown in Figs. 3 – 5 and compare the resulting computations with the relative illumination plots provided in those figures.

In all three figures, there are no lens elements between the iris and the object, so the ratio $[A_{EN}(y)/A_{EN}(0)]$ is equal to 1 (i.e., the iris in this case is the entrance pupil). Applying the lens system conditions in Fig. 3 to an optical design program will yield $\theta = 21.644^\circ$, $D = -0.145$, and $dD/dy = -0.002907 \text{ mm}^{-1}$. Inserting these values in Eq. (1), we have $R(y) \approx 1.55$ at the full height of the object. According to Fig. 3b, the relative illumination from the plot reads $\approx 1/0.75 \approx 1.33$, so our computation is somewhat close to this value (note that the relative illumination in the plots is always normalized to the peak value).

Applying the conditions from Fig. 4 into the ray tracing program I am using, I obtain $\theta = 20.964^\circ$, $D = -0.066$, and $dD/dy = -0.001249 \text{ mm}^{-1}$. Inserting these values into Eq. (1) yields $R(y) \approx 1.01$, which seems close to the maximum value for relative illumination shown in Fig. 4b. Finally, the condition in Fig. 5 results in $\theta = 19.654^\circ$, $D = -0.033$, and $dD/dy = -0.0006312 \text{ mm}^{-1}$. Inserting these values into Eq. (1) yields $R(y) \approx 0.9$, which is close to the relative illumination at the full object height shown in Fig. 5b.

Thus, the equation works pretty well, proving that relative illumination may be viewed as a function of the ray geometry at either the entrance pupil or the exit pupil. Moreover, it is quite evident that differential distortion dD/dy plays a significant role in the relative illumination of a lens system.

Relative Illumination In Projection And Machine Vision

The relative illumination in projection and machine vision systems generally is a more complex function of lens and system variables. When designing the illumination subsystem in, say, a projector, one often applies optical components, such as condensers, light pipes, and lens arrays (also known as “fly's eye lenses”). A condenser serves to collect as much flux as possible from a source, while the light pipes and lens arrays serve to homogenize the illumination distribution across the projection target plane.

In some other cases of illumination design, the surfaces of optical components may take on any shape, and such “freeform” lenses and mirrors can produce virtually any desired irradiance distribution across a screen. The result is that relative illumination for such systems cannot be described by Eq. (1), nor is it a simple function of the exit pupil's ray geometry. Rather, the irradiance distribution across a screen is best described using the methods of illumination engineering and nonimaging optics.^{14–17}

In machine vision systems, lenses form images onto electronic array sensors, such as CMOS image sensors, whose pixels often have microlenses assembled in front of them. Consequently, light rays in such systems can experience clipping by the microlenses at the edge of the field of view, unless the imaging lens' oblique ray angles have been designed to match specific geometries of the microlenses.¹⁸

Additionally, in machine vision systems, one generally mounts light sources next to imaging lenses to illuminate the region of interest for the vision system. A “ring” of LEDs, for example, may be mounted surrounding the camera lens, and the LED arrangement can be made to balance the inherent relative illumination distribution given by the lens design. Therefore, the final irradiance distribution at the image can be tuned by a virtually unlimited arrangement of illumination geometries.

Where Does The Cosine Fourth Power Law Apply?

There is but one condition in which the cosine fourth power law applies. In the subject of optical radiometry, one learns that the irradiance across a screen from a small Lambertian source does, indeed, follow a fourth-power cosine rule (Fig. 7). By “small” it is meant that the surface area of the source is significantly less than the source's distance to the screen. Under this condition, the irradiance $E(\theta)$ at the screen is well-described by the following equation:

$$E(\theta) \approx \frac{LA \cos^4 \theta}{z^2} \quad \text{Eq. (2)}$$

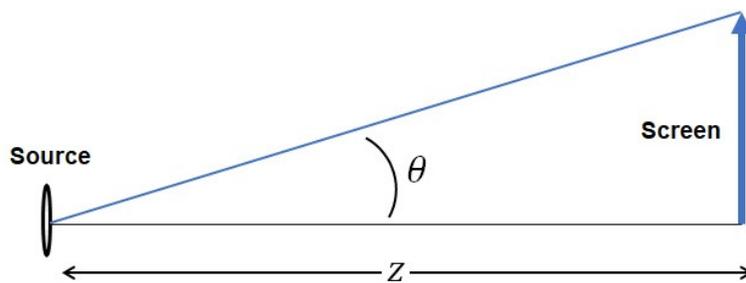
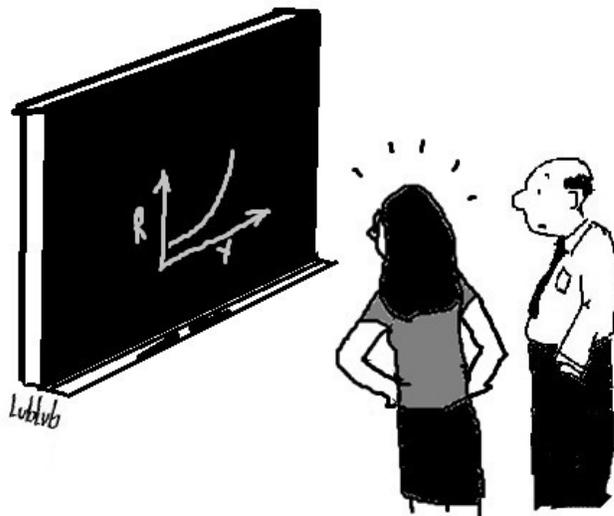


Fig. 7 — Small Lambertian source illuminating a distant screen.

In Eq. (2), L is the source's radiance, and A is the source's surface area. Since it is known that the quantity L in Eq. (2) is invariant in a lossless optical system, it has been said that the exit pupil of an imaging lens appears as a Lambertian source from the perspective of the image.¹⁹ This is quite accurate, as long as we think of the exit pupil as a Lambertian source of varying shape and size as

it is viewed from across the image plane. Unfortunately, this has, at times, also led to the misconception that the relative illumination across the image plane would be described by Eq. (2).

But, as we have seen in this article, this simple cosine rule does not generally apply to lens systems. Rather, the relative illumination of a lens system is a function of the source's emission characteristics, the shape and size of the entrance and exit pupils, the obliquity of rays relative to the optic axis, the distortion in the image, and the differential distortion across the image.



"Well, that's been relatively illuminating, wouldn't you say?"

About The Author

Ronian Siew [is a consultant](#) in optical engineering. His optical designs have included objectives, eyepieces, relays, fluorescence detection systems for DNA analysis, and more. He has authored two books and a number of scientific peer-reviewed and conference papers. Ronian holds a MS in Optics and a degree in Physics from the University of Rochester. He may be reached at ronian@inopticalsolutions.com.

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